Analysis Methods in Atmospheric and Oceanic Science

AOSC 652

Partial Differential Equations

Week 13, Day 1

21 Nov 2016

AOSC 652: Analysis Methods in AOSC

Partial Differential Equations

Classic PDE that describes the flow of heat along a pipe:

$$\frac{\partial u}{\partial t}(x,t) = K\frac{\partial^2 u}{\partial x^2}(x,t) \qquad (1)$$

Alternate forms of this equation:

$$c\rho\frac{\partial u}{\partial t}(x,y,z,t) = \frac{\partial u}{\partial x}\left(k(x,y,z)\frac{\partial u}{\partial x}(x,y,z,t)\right) + \frac{\partial u}{\partial y}\left(k(x,y,z)\frac{\partial u}{\partial y}(x,y,z,t)\right)$$
$$+\frac{\partial u}{\partial z}\left(k(x,y,z)\frac{\partial u}{\partial z}(x,y,z,t)\right) \qquad (2)$$

where *c* is specific heat
ρ is density
and *k* is thermal conductivity

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$$+\frac{\partial u}{\partial z}\left(k(x,y,z)\frac{\partial u}{\partial z}(x,y,z,t)\right) \qquad (2)$$

Equation can be re-written (and solved in a straight forward manner) if *c*, ρ, and *k* are constant wrt spatial dimensions:

$$\frac{c\rho}{k}\frac{\partial u}{\partial t}(x,y,z,t) = \frac{\partial^2 u}{\partial x^2}(x,y,z,t) + \frac{\partial^2 u}{\partial y^2}(x,y,z,t) + \frac{\partial^2 u}{\partial z^2}(x,y,z,t) \qquad (3)$$

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Partial Differential Equations

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How might a physicist write this same equation?

where T is temperature,

∇T is the gradient of temperature

$\nabla\bullet$ is the divergence operator

ρ is the density

C_p is the heat capacity

Q is the heat added to the system

and k is the thermal conductivity

Partial Differential Equations

Classic PDE that describes the flow of heat along a pipe:

$$\frac{\partial u}{\partial t}(x,t) = K\frac{\partial^2 u}{\partial x^2}(x,t) \qquad (1)$$

Another application of this same equation in AOSC that does not involve Temperature or Heat:

$$\frac{\partial \text{ Carbon}}{\partial t}(\text{z},t) = K\frac{\partial^2 \text{Carbon}}{\partial z^2}(z,t) \qquad (5)$$

where Carbon is the Dissolved Inorganic Carbon content of the ocean

z is the depth of the ocean

K is the eddy diffusion coefficient

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Partial Differential Equations

A classic example of PDEs from oceanography:

For the barotropic, tropical ocean, the linear inviscid governing equations (Gill, 1982) are:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \beta y v = -g\dfrac{\partial \eta}{\partial x} & \text{(6a)} \\ \dfrac{\partial v}{\partial t} + \beta y u = -g\dfrac{\partial \eta}{\partial y} & \text{(6b)} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = -\dfrac{1}{H}\dfrac{\partial \eta}{\partial t} & \text{(6c)} \end{cases}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \beta y\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \beta v = 0 \quad \text{(6d)}$$

Partial Differential Equations

A classic example of PDEs from atmospheric sciences:

For the barotropic atmosphere the governing equations (Anderson and McCreary, 1985) are:

$$\begin{cases} \dfrac{\partial U}{\partial t} - \beta y V = -\dfrac{\partial P}{\partial x} - rU & \text{(7a)} \\ \dfrac{\partial V}{\partial t} + \beta y U = -\dfrac{\partial P}{\partial y} - rV & \text{(7b)} \\ \dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y} = -r\dfrac{P}{c^2} & \text{(7c)} \end{cases}$$

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Partial Differential Equations

Coupled ocean/**atmosphere** system:

The ocean

$$\begin{cases} \dfrac{\partial u}{\partial t} - \beta y v = -g\dfrac{\partial \eta}{\partial x} + \dfrac{\tau_x}{\rho_0 H}, \\ \dfrac{\partial v}{\partial t} + \beta y u = -g\dfrac{\partial \eta}{\partial y}, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = -\dfrac{1}{H}\dfrac{\partial \eta}{\partial t}, \end{cases}$$

The atmosphere

$$\begin{cases} \dfrac{\partial U}{\partial t} - \beta y V = -\dfrac{\partial P}{\partial x} - rU, \\ \dfrac{\partial V}{\partial t} + \beta y U = -\dfrac{\partial P}{\partial y} - rV, \\ \dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y} = -r\dfrac{P}{c^2} + \dfrac{Q}{c^2}, \end{cases}$$

Partial Differential Equations

One more example of a classic PDE from AOSC:

Potential Vorticity: $$PV = -g\ (\zeta_\theta + f)\ \frac{\partial \theta}{\partial p} \qquad \text{(8a)}$$

where:

PV is potential vorticity

f is Coriolis Parameter, 2 Ω sin(Latitude), where Ω is angular speed of Earth's rotation (s^{-1})

g is gravitational acceleration ($m\ s^{-2}$)

p is pressure (mbar, or force/area ⇨ $kg\ m\ s^{-2} / m^2$)

θ is potential temperature (K)

ζ_θ is relative isentropic vorticity (s^{-1})

$$\zeta_\theta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \text{ evaluated on an isentropic surface} \qquad \text{(8b)}$$

Units: $K\ m^2\ s^{-1}\ kg^{-1}$. The quantity $10^{-6}\ K\ m^2\ s^{-1}\ kg^{-1}$ is often called MKS PV units

Numerical Solutions to Partial Differential Equations

Classic PDE that describes the flow of heat along a pipe:

$$\frac{\partial u}{\partial t}(x,t) = K\frac{\partial^2 u}{\partial x^2}(x,t) \qquad (1)$$

Typically in atmospheric and oceanic sciences, PDEs are solved *numerically* rather than *analytically*:

although in many cases *analytic manipulation* is *key* to understanding the solutions (e.g., wave propagation) or even being able to proceed with a numerical solution

Three aspects of Numerical Solutions to PDEs that must be considered:

1. Initial Conditions and Boundary Conditions
2. Numerical Stability
3. Numerical Diffusion

Note: material presented in this part of the lecture is based on Chapter 12 (Numerical Solutions to PDEs) of Numerical Analysis, 8th Edition, by Richard L. Burden and J. Douglass Faires

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Numerical Solutions to Partial Differential Equations

Classic PDE that describes the flow of heat along a pipe:

$$\frac{\partial u}{\partial t}(x,t) = K\frac{\partial^2 u}{\partial x^2}(x,t) \tag{1}$$

1. Initial Conditions (IC) and Boundary Conditions (BC)

For the heat diffusion equation, must typically specify distribution of temperature (or heat) at all locations for *t = 0*

$$u(x,0) = f(x) \qquad \text{(IC)}$$

Also, must specify either value of temperature (or heat) at boundary:

$$u(0,t) = u_0 \quad \text{and} \quad u(L,t) = u_L \quad \text{where the rod has length } L \qquad \text{(BCa)}$$

or flux of temperature (or heat) at boundary:

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0 \quad \text{if the rod is insulated on both ends} \qquad \text{(BCb)}$$

Let's examine how our solution depends on IC and BC !

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Numerical Solutions to Partial Differential Equations

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Let's examine how our solution depends on IC and BC !

Can show that for BCa $\lim_{t \to \infty} u(x,t) = u_0 + x\,\frac{u_L - u_0}{L}$

Can show that for BCb $\lim_{t \to \infty} u(x,t) = \text{Constant}$

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Numerical Solutions to Partial Differential Equations

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$$\frac{\partial u}{\partial t}(x,t) = K\frac{\partial^2 u}{\partial x^2}(x,t) \qquad (1)$$

$$u(0,t) = u(L,t) = 0, \quad t > 0 \quad \text{and} \quad u(x,0) = f(x), \quad 0 \le x \le L \qquad \text{(IC) \& (BCa)}$$

2. Numerical Stability

Select a time-step size *k*

$$\frac{\partial u}{\partial t}(x_i,t_j) \approx \frac{u(x_i,t_j) - u(x_i,t_{j-1})}{k} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \ldots$$

Select a spatial-step size *h*

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) \approx \frac{u(x_i+h,t_j) - 2u(x_i,t_j) + u(x_i-h,t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x4} + \ldots$$

Then:

$$\frac{u_{i,j} - u_{i,j-1}}{k} - K\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0$$

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<u>Numerical Solutions</u> to Partial Differential Equations

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2. Numerical Stability

The relation:

$$\frac{u_{i,j} - u_{i,j-1}}{k} - K\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0$$

combined with the IC & BC imply:

$$\begin{bmatrix} (1+2\lambda) & -\lambda & 0 & \cdots & 0 \\ -\lambda & (1+2\lambda) & -\lambda & \cdots & \vdots \\ \vdots & -\lambda & (1+2\lambda) & -\lambda & \vdots \\ \vdots & \cdots & -\lambda & \ddots & -\lambda \\ 0 & \cdots & \cdots & -\lambda & (1+2\lambda) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{m-1,j} \end{bmatrix} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ \vdots \\ u_{m-1,j-1} \end{bmatrix}$$

$$\text{where } \lambda = K(k/h^2)$$

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Numerical Solutions to Partial Differential Equations

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The relation:

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Another matrix equation, finally !

where $\lambda = K(k/h^2)$

Numerical Solutions to Partial Differential Equations

2. Numerical Stability has "everything to do" with the properties of the matrix !

It can be shown that, due to the properties of the eigenvalues of this matrix, it's inverse must exist: therefore, can always find values of *u* at time *j* based on knowledge of values at time *j* – *1*

This solution ***backward difference*** or ***implicit soln*** is said to be **unconditionally stable**

$$\begin{bmatrix} (1+2\lambda) & -\lambda & 0 & \cdots & 0 \\ -\lambda & (1+2\lambda) & -\lambda & \cdots & \vdots \\ \vdots & -\lambda & (1+2\lambda) & -\lambda & \vdots \\ \vdots & \cdots & -\lambda & \ddots & -\lambda \\ 0 & \cdots & \cdots & -\lambda & (1+2\lambda) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{m-1,j} \end{bmatrix} = \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ \vdots \\ u_{m-1,j-1} \end{bmatrix}$$

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Numerical Solutions to Partial Differential Equations

2. **Numerical Stability** has "everything to do" with the properties of the matrix !
It can be shown that, due to the properties of the eigenvalues of this matrix, it's inverse must exist: therefore, can always find values of *u* at time *j* based on knowledge of values at time *j* – *1*

This solution ***backward difference*** or ***implicit soln*** is said to be **unconditionally stable**

Interestingly, had we defined:

$$\frac{\partial u}{\partial t}(x_i, t_j) \approx \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \ldots$$

we would have arrived at a different matrix that can only be inverted when *k*, *h*, and *K* satisfy a specific relation. This solution, called the ***forward difference*** or ***explicit soln*** is said to be **conditionally stable**

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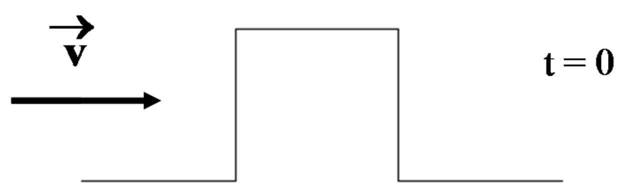
Numerical Solutions to Partial Differential Equations

3. Numerical Diffusion

Consider the continuity equation:

$$\frac{\partial\ Chemical}{\partial t} = v \cdot \Delta\ (Chemical) + Production - Loss$$

Let's examine the advective term & consider transport of a square wave to the right with velocity v:



Move the material 0.1 grid box to the right:

Note: material presented in this part of the lecture is based on an excellent presentation by Anne Douglass (NASA GSFC) that was given at the University of Toronto, available on the web at: http://www.atmosp.physics.utoronto.ca/MAM/douglass2.ppt

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<u>Numerical Solutions</u> to Partial Differential Equations

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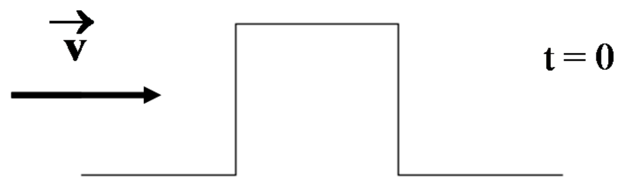

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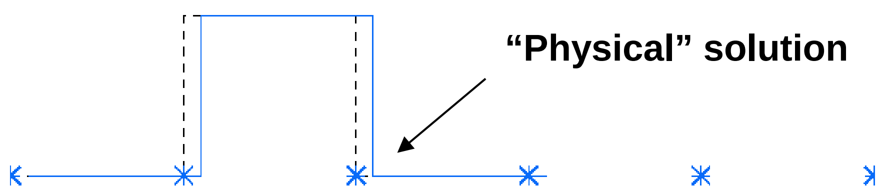
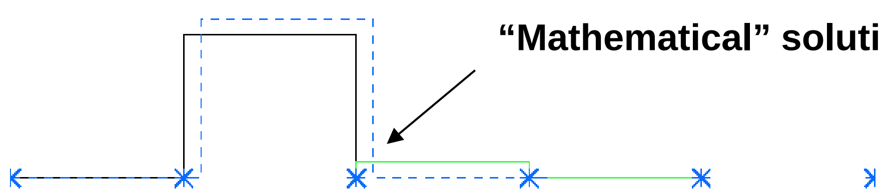


<u>Critical aspect: what happens at next time step!</u>

Upwind material mixes uniformly in neighboring grid box.



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<u>Numerical Solutions</u> to Partial Differential Equations

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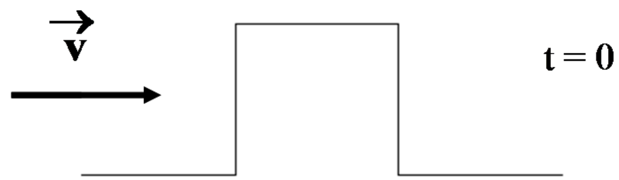

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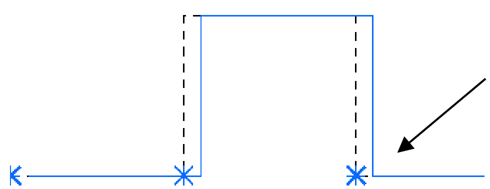
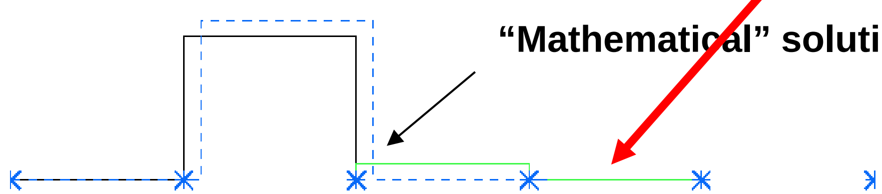


"Physical" solution

***<u>Next time step</u>*: some of the material will appear here, which disagrees with physical solution (i.e., non-physical)**

Upwind material mixes uniformly in neighboring grid box.



"Mathematical" solution

Numerical Solutions to Partial Differential Equations

3. Numerical Diffusion

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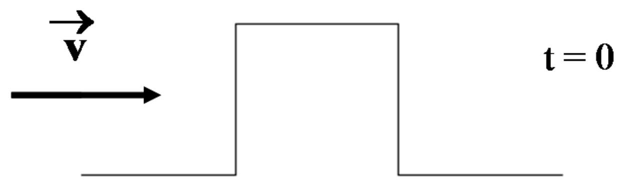

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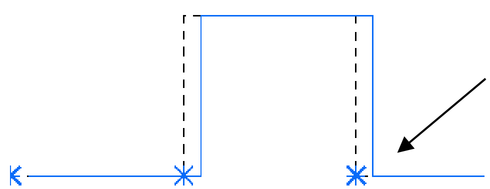

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NUMERICAL DIFFUSION

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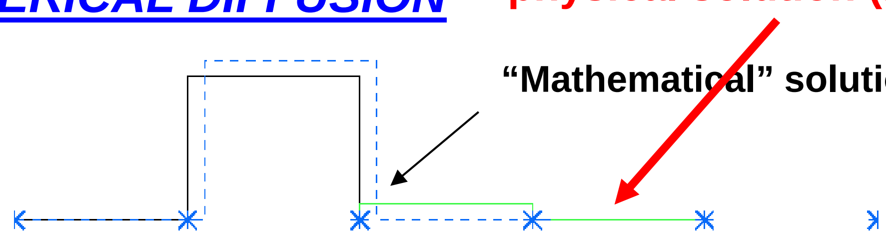

"Mathematical" solution

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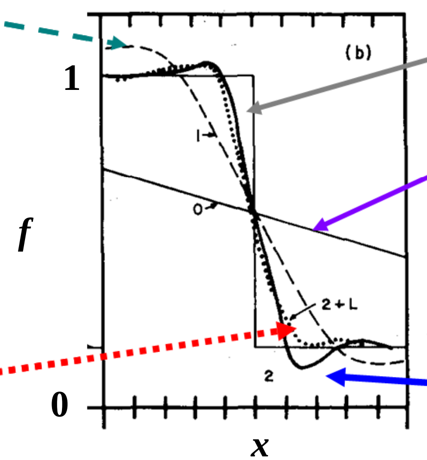
Prather, M. J., Numerical advection by conservation of second-order moments, *JGR, 91*, 6671-6681, 1986 developed an algorithm that almost exactly accounts for the advective transport of a constituent in grid box of appropriate size of 3D climate model

Advantages: Non-diffusive, mass-conserving, numerically stable



"slopes" first order moments (overshoot and undershoot)!

start with a step function

upwind (very diffusive)

second order moments with limiter (no undershoot)

second order moments (small overshoot and undershoot)

Disadvantage: Computational requirements (need nine 3D arrays for each constituent)

Take home message: if conducting 3D calculations of the distribution of a constituent that involves the solution of a PDE, will often have to consider (and account for) effects of numerical diffusion (there are numerous schemes to consider)