Time series: Frequency domain

We will discuss here first Fourier transform and spectra, and then time filters.

Fourier series for continuous data

Assume we have data $f(t)$ in a time interval $\left[0,T\right]$. Then the data can be expressed as a Fourier series

$$f(t)=\frac{A_0}{2}+\sum_{k=1}^{\infty}\left[A_k\cos\left(k\frac{2\pi}{T}t\right)+B_k\sin\left(k\frac{2\pi}{T}t\right)\right](1),$$

or, using an exponential form,

$$f(t)=\sum_{k=-\infty}^{\infty}C_k\exp\left(ik\frac{2\pi}{T}t\right).$$

Here $\nu_0=\frac{2\pi}{T}$ is the fundamental frequency (corresponding to a period *T*).

$\nu_k=\frac{2\pi k}{T}=\nu_0 k$ are the higher frequencies (harmonics).

The Fourier transforms of $f(t)$ can be obtained by multiplying (1) by $\cos\left(k\frac{2\pi}{T}t\right)$ or $\sin\left(k\frac{2\pi}{T}t\right)$, integrating over $\left[0,T\right]$ and using $\int_0^T\cos^2(k\frac{2\pi}{T}t)dt=\frac{1}{2}\int_0^T\left[\cos^2(k\frac{2\pi}{T}t)+\sin^2(k\frac{2\pi}{T}t)\right]dt=\frac{1}{2}\int_0^T dt=\frac{T}{2}$. We then get

$$A_k=\frac{2}{T}\int_0^T f(t)\cos\left(k\frac{2\pi}{T}t\right)dt$$
$$B_k=\frac{2}{T}\int_0^T f(t)\sin\left(k\frac{2\pi}{T}t\right)dt,$$

or, in exponential form

$$C_k=\frac{1}{T}\int_0^T f(t)\exp\left(-ik\frac{2\pi}{T}t\right)dt$$

The sine/cosine and exponential coefficients are related by

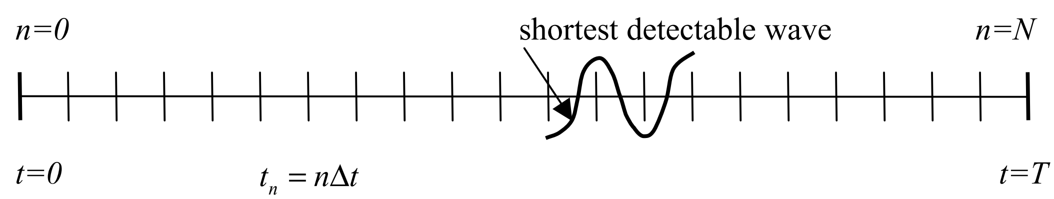
$$C_k = \begin{Bmatrix} \frac{1}{2}(A_k - iB_k) \ \text{ for } k \geq 0 \\ \frac{1}{2}(A_{-k} + iB_{-k}) \ \text{ for } k < 0 \end{Bmatrix}.$$

$\frac{A_0}{2} = \frac{1}{T}\int_0^T f(t)dt = \overline{f(t)}$ corresponds to the frequency $\nu = 0$.

The power spectrum of a time series is the square of the amplitude of each harmonic, and it provides the contribution of each harmonic to the total energy of the time series $\overline{f'^2(t)}$:

$$P_k^2 = A_k^2 + B_k^2$$

Now, if we have a discrete time series of length N, with equal time intervals Δt, the formulas are similar, with $t_n = n\Delta t; \quad T = N\Delta t$, and $N+1$ distinct Fourier coefficients ($N+1$ is the number of discrete points in the series, since $n=0,1,\ldots,N$).



$$f_n = f(t_n) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} C_k \exp\left(ik\frac{2\pi}{N\Delta t}n\Delta t\right) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} C_k \exp\left(i\frac{2\pi k}{N}n\right)$$

Or, in terms of sines and cosines,

$$f(t_n) = \frac{A_0}{2} + \sum_{k=1}^{\frac{N}{2}} \left[A_k \cos\left(\frac{2\pi k}{N} n\right) + B_k \sin\left(\frac{2\pi k}{N} n\right) \right]$$

The coefficients are obtained, as before, from the discrete Fourier transform that gives the amplitude of the signal due to each wave number or harmonic:

$$A_k = \frac{2}{N} \sum_{n=1}^{N} \left[f_n \cos\left(\frac{2\pi k}{N} n\right) \right]$$

$$B_k = \frac{2}{N} \sum_{n=1}^{N} \left[f_n \sin\left(\frac{2\pi k}{N} n\right) \right]$$

or, in complex exponential form

$$C_k = \frac{1}{N} \sum_{n=0}^{N} \left[f_n \exp\left[-i\left(\frac{2\pi k}{N} n\right) \right] \right]$$

with power spectrum

$$P_k^2 = A_k^2 + B_k^2 = C_k^2$$

Parseval's theorem: the average energy is the same in physical or Fourier space:

$$\frac{1}{T} \int_0^T f(t_n)^2 dt = \frac{A_0^2}{2} + \sum_{k=1}^{\frac{N}{2}} \left[A_k^2 + B_k^2 \right]$$

Example: Annual cycle

Assume we have monthly means, and we average all years to obtain $\bar{f}_n$, monthly averages corresponding to each month of the year. Δt is then one month, and $N=12$. Typically, we can represent the annual cycle with at least two harmonics, to allow for a lack of symmetry between winter and summer:

$$f_{ACn} = \frac{A_0}{2} + A_1 \cos\left(\frac{2\pi}{12/1}n\right) + B_1 \sin\left(\frac{2\pi}{12/1}n\right) + A_2 \cos\left(\frac{2\pi}{12/2}n\right) + B_2 \sin\left(\frac{2\pi}{12/2}n\right)$$

The A_0 term represents the annual average (with zero frequency), the A_1 and B_1 terms represent the periodic component with period 12 months (fundamental frequency), and the A_2 and B_2 terms represent a periodic component with period 6 months (first harmonic).

The coefficients A_0, A_1, A_2, B_1, B_2 can be obtained as before, e.g.,

$B_2 = \frac{2}{12}\sum_{n=1}^{12} \bar{f}_n \sin\left(\frac{2\pi}{12/2}n\right)$, where the bar represents the monthly average over several years.

Once the coefficients are obtained, the annual cycle can be subtracted from the time series in order to deal with *anomalies*.