

AOSC 614 – Addition to Section 3.2.3 (page 82)

Robert-Asselin-Williams (RAW) filter for the Leap-frog scheme

Paul Williams (2009) has developed a significant improvement to the Robert filter: Fig. 1a from his paper shows schematically the standard Robert filter for the equation $\frac{\partial U}{\partial t} = F(U)$ using the leap-frog scheme

$$U_{n+1} = \overline{U_{n-1}} + 2\Delta t \cdot F(U^n)$$

followed by an uncentered time filter

$$\overline{U}_n = U_n + \frac{\delta}{2} [U_{n+1} - 2U_n + \overline{U_{n-1}}]$$

In the Robert time filter, \overline{U}_n is displaced from the original unfiltered value of U_n towards the average of $\overline{U_{n-1}}$ and U_{n+1} a

distance $d = \frac{\delta}{2} [U_{n+1} - 2U_n + \overline{U_{n-1}}]$. The Robert filter reduces by

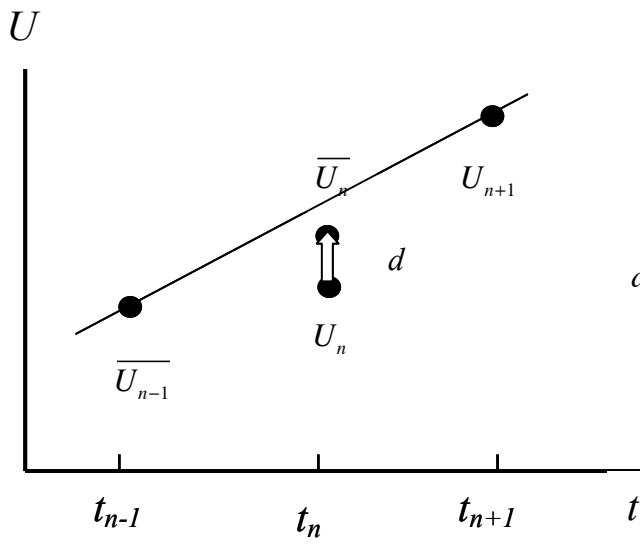
$(1 - \delta)$ the “second derivative” or “curvature” of the three points,

$|U_{n+1} - 2\overline{U}_n + \overline{U_{n-1}}| = (1 - \delta) |U_{n+1} - 2U_n + \overline{U_{n-1}}|$ as desired, and this

“smoothing” strongly damps the computational mode, which is the purpose of the filter. Unfortunately, applying the filter also changes the mean value from its original mean value before applying the time filter:

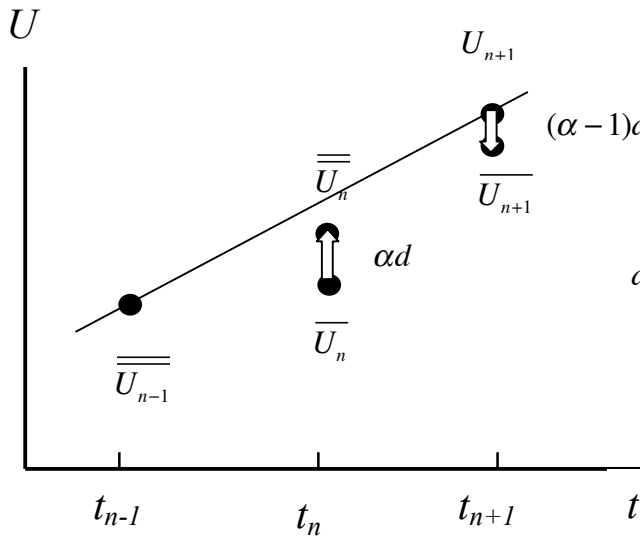
$$M_{RA} = (\overline{U_{n-1}} + \overline{U}_n + U_{n+1}) = (\overline{U_{n-1}} + U_n + U_{n+1}) + d.$$

As a result the Robert filter introduces first order truncation errors (albeit multiplied by the small parameter δ) and significantly damps the amplitude of the physical wave.



Standard Robert-Asselin Filter

$$d = \frac{\delta}{2} [U_{n+1} - 2U_n + \overline{U}_{n-1}]$$



New Robert-Asselin Williams Filter

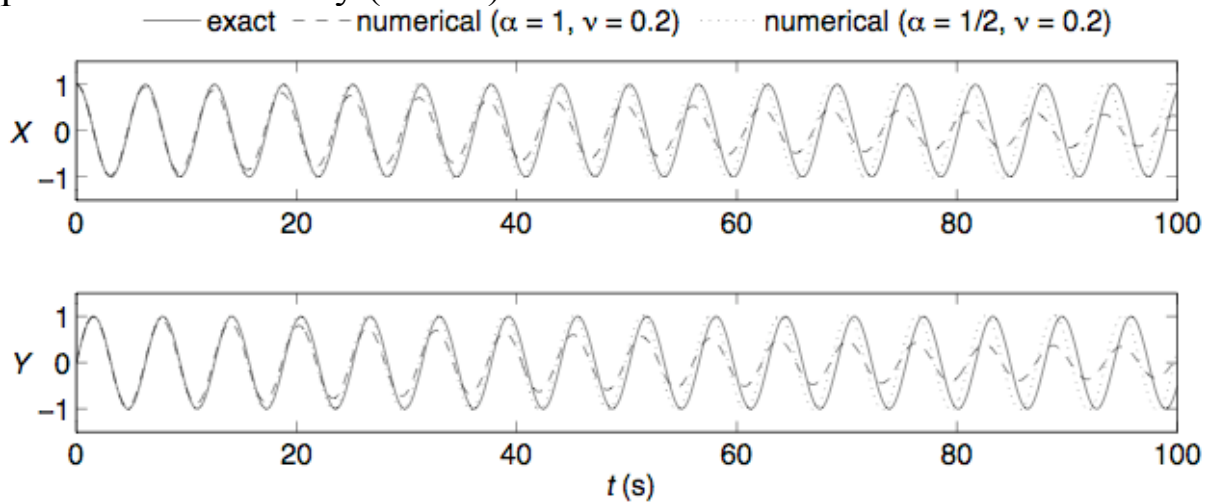
$$d = \frac{\delta}{2} [U_{n+1} - 2\overline{U}_n + \overline{\overline{U}}_{n-1}]$$

In order to ameliorate this problem, Williams proposes a new filter, shown in his schematic Fig. 1b, in which $\overline{\overline{U}}_n$ is displaced from \overline{U}_n towards the average of U_{n+1} and $\overline{\overline{U}}_{n-1}$ a distance αd , where $d = \frac{\delta}{2}(U_{n+1} - 2\overline{U}_n + \overline{\overline{U}}_{n-1})$, and at the same time U_{n+1} is displaced a distance $(\alpha - 1)d$, with $0 \leq \alpha \leq 1$.

Now the mean value after applying the RAW filter is

$$M = (\overline{\overline{U}}_{n-1} + \overline{\overline{U}}_n + \overline{\overline{U}}_{n+1}) = (\overline{U}_{n-1} + \overline{U}_n + U_{n+1}) + (2\alpha - 1)d,$$

If $\alpha = 1/2$ then the mean is conserved exactly, while still reducing the curvature. Figure 2 from Williams (2009) shows that the application of the Robert filter ($\alpha = 1$) damps the physical solution (dashed), whereas the choice $\alpha = 1/2$ maintains its amplitude and phase almost exactly (dotted).



Exact solution (full line) and two numerical solutions to the wave equation X and Y components, Robert-Asselin (RA) filter with dashes and Robert-Asselin-Williams (RAW) filter with dots.

Williams recommends the use of a controlling parameter α slightly larger than $\frac{1}{2}$, e.g., $\alpha = 0.53$. This has the effect of keeping the amplification factor of the filter smaller but very close to one, whereas the original Robert Filter, corresponding to $\alpha = 1$, has a much stronger damping effect (Fig. 5 from Williams 2009). The “curvature” is reduced as much as in the original RA filter, so the RAW filter still damps the computational mode.

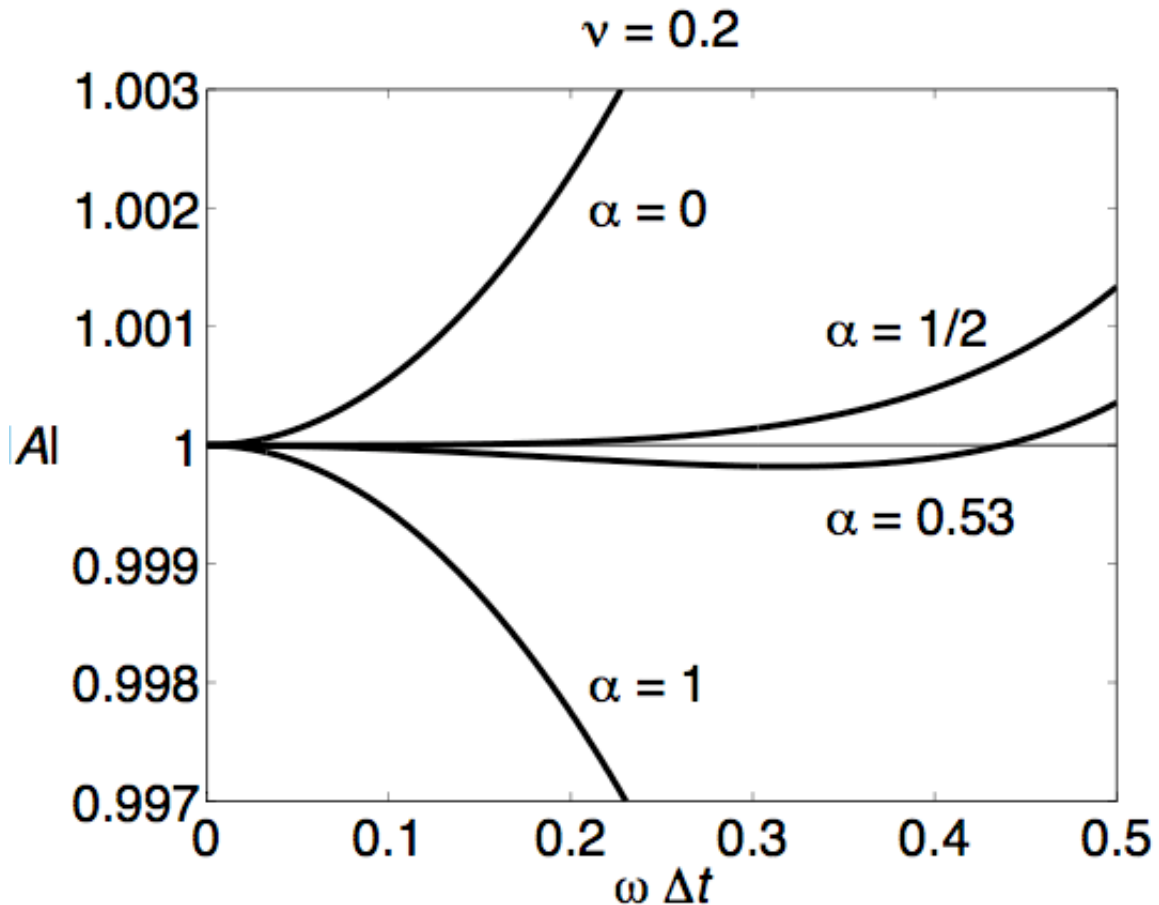


Figure 5:

The steps in the RAW filter are then the standard leap-frog scheme which starts with $\overline{\overline{U}}_{n-1}$ having been modified twice in the previous two time steps, and \overline{U}_n modified once in the previous time step:

$$U_{n+1} = \overline{\overline{U}}_{n-1} + 2\Delta t \cdot F(\overline{U}_n),$$

followed by two filtering steps:

$$\overline{\overline{U}}_n = \overline{U}_n + \frac{\delta\alpha}{2} \left[\overline{\overline{U}}_{n-1} - 2\overline{U}_n + U_{n+1} \right]$$

$$\overline{U}_{n+1} = U_{n+1} + \frac{\delta(\alpha-1)}{2} \left[\overline{\overline{U}}_{n-1} - 2\overline{U}_n + U_{n+1} \right]$$

The filtered values immediately replace the previous values so that there is no memory overhead.

Williams shows that the unfiltered, singly filtered and doubly filtered values share a common complex amplification factor

$$A = \frac{U_{n+1}}{U_n} = \frac{\overline{U}_{n+1}}{\overline{U}_n} = \frac{\overline{\overline{U}}_{n+1}}{\overline{\overline{U}}_n}$$

Figure 5 from Williams (2009) plots the amplitude of $|A|$ for different values of α . The advantage of the RAW filter with the suggested value $\alpha = 0.53$ is evident.

Exercise (Homework 2)

1) From Figure 5 in Williams (2009) discuss the properties (maximum allowable time step for a given computational frequency ω , i.e., $|\omega\Delta t| \leq 1$) of a filter with $\alpha = 0$, where only the new time step $n+1$ value is filtered, $\alpha = 0.5$, $\alpha = 0.53$, the values close to optimal in the RAW filter, and $\alpha = 1$, corresponding to the standard Robert filter.

2) Program in MatLab or FORTRAN the coupled inertia wave equations, with f the Coriolis parameter.

$$\frac{du}{dt} = fv$$
$$\frac{dv}{dt} = -fu$$

using leap-frog and computing the RHS exactly at time n .

Explore the stability for different values of $f\Delta t$ (the “Courant number”). Plot the results (u, v) versus time.

3) Implement the Robert filter and the RAW filter.

4) Implement a wave/dissipation equation

$$\frac{du}{dt} = fv - \sigma u$$

$$\frac{dv}{dt} = -fu - \sigma v$$

and explore the results using a leap-frog, forward (with two time steps), and implicit centered and backward scheme for the diffusion terms for different diffusion values. Discuss the results. Note: here the Coriolis term is “hyperbolic-like”, since it has a real frequency f (oscillatory, like waves), and the damping term is “parabolic-like”, since it has a real negative exponent, like diffusion. Therefore the Coriolis term can be written with Leapfrog, and the damping term with a forward time step ($2*dt$) when combined with leap-frog). Or one can use a centered implicit scheme for both since it is stable.

5) Examine the FORTRAN code of the SPEEDY model and show that it uses a leap-frog scheme with a Robert filter for the “wave-like” terms and a two-time-step forward scheme for the diffusion scheme.

6) (Optional) Modify the Robert filter into a RAW filter. Explore the impact of the changed filter in the 2-day forecast and on the monthly averaged forecasts (e.g., 500hPa height, precipitation).