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Is the Isentropic Surface Always Impermeable to the Potential Vorticity Substance?

Chanh Q. KIEU*[1] and Da-Lin ZHANG[2]

[1] *Laboratory of Climate and Weather Research,*

Hanoi College of Science, Vietnam National University, Vietnam 10000

[2] *Department of Atmospheric and Oceanic Science, University of Maryland, College Park, Maryland* 20740

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ABSTRACT

The impermeability of isentropic surfaces by the potential vorticity substance (PVS) has often been used to help understand the generation of potential vorticity in the presence of diabatic heating and friction. In this study, we examined singularities of isentropic surfaces that may develop in the presence of diabatic heating and the fictitious movements of the isentropic surfaces that are involved in deriving the PVS impermeability theorem. Our results show that such singularities could occur in the upper troposphere as a result of intense convective-scale motion, at the cloud top due to radiative cooling, or within the well-mixed boundary layer. These locally ill-defined conditions allow PVS to penetrate across an isentropic surface. We conclude that the PVS impermeability theorem is generally valid for the stably stratified atmosphere in the absence of diabatic heating.

Key words: PV substance impermeability, singularities of isentropic surfaces, diabatic heating

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1. Introduction

Potential vorticity (PV) is an important dynamical variable due to its conservative property in the absence of diabatic heating and friction and to its invertibility principle that would allow one to obtain the three-dimensional (3D) structures of balanced flows (Hoskins et al., 1985; Davis and Emanuel, 1991; Huo et al., 1999; Zhang et al., 2002; Kieu and Zhang, 2010). Haynes and McIntyre (1987, hereafter referred to as HM87), extended Hoskins et al. (1985)'s work and presented the most complete and diligent treatment of the PV concept, particularly from the point of view of PV substance (PVS), defined as ρQ, where ρ is air density and Q is PV. The work presented in HM87 proved to be a powerful theorem, later called the PVS impermeability theorem, containing two general but important conclusions for the stably stratified atmosphere in the presence of arbitrary diabatic heating and friction:

(1) There is no net transport of PVS across any isentropic surface.

(2) PVS can never be created or destroyed within a layer bounded by two isentropic surfaces.

As pointed out by Danielsen (1990), there is some confusion in the general impermeability theorem presented in HM87 between PV and PVS since it is often PV, not PVS, that is of the main interest to meteorological analyses. PV is related to balanced dynamics, and its invertibility principle allows the balanced dynamics associated with the PV to be extracted, especially in the absence of diabatic heating and frictional forcings (see Davis and Emanuel, 1991; Zhang and Kieu, 2006; Egger, 2008).

In this paper, we show that the movement of isentropic surfaces, as argued in HM87 and later refined and presented in the work of Haynes and McIntyre (1990, hereafter referred to as HM90) is not only potentially ill-defined in the presence of diabatic heating (or cooling) but also inapplicable in the real atmosphere where deep convection occurs. We further assert that the PVS impermeability theorem is only valid in the absence of diabatic heating.

*Corresponding author: Chanh Q. KIEU, chanhkq@vnu.edu.vn

2. Review

HM90 presented the first derivation of the flux form of PV equation in the Cartesian coordinates as follows:

$$\frac{\partial(\rho Q)}{\partial t} + \nabla \cdot (\rho Q \boldsymbol{u} - H\boldsymbol{\zeta} - \boldsymbol{F} \times \nabla\theta) = 0\ , \qquad (1)$$

where $H = d\theta/dt$ is the diabatic heating rate, $\boldsymbol{u} = (u, v, w)$ is the 3D flow field, $\boldsymbol{\zeta}$ is the 3D absolute vorticity, θ is the potential temperature, and $\boldsymbol{F}$ denotes frictional effects. Upon introducing a control volume V bounded by two time-dependent isentropic surfaces ∂V_θ at the top and bo ttom and lateral boundaries ∂V_s, and integrating over the volume V, in HM90, Haynes and McIntyre (1990) obtained the integral form of the PV equation, i.e., Eq. (4.5) of HM87:

$$\begin{aligned}\frac{d}{dt}\int_V \rho Q dV = &-\int_{\partial \boldsymbol{V}_\theta} [(\boldsymbol{u}-\boldsymbol{U})\rho Q - H\boldsymbol{\zeta} - \\ &\boldsymbol{F}\times\nabla\theta]\cdot \boldsymbol{n} dS - \\ &\int_{\partial V_s} [(\boldsymbol{u}-\boldsymbol{U})\rho Q - H\boldsymbol{\zeta} - \\ &\boldsymbol{F}\times\nabla\theta]\cdot \boldsymbol{n} dS\ ,\end{aligned} \qquad (2)$$

where $\boldsymbol{U} = (U, V, W)$ is the 3D velocity of the boundaries. We are henceforth concerned only with the vertical fluxes across the top and bottom surfaces ∂V_θ, i.e., the first surface integral on the right hand side (rhs) of Eq. (2), hereafter referred to as VFX. In HM90, the impermeability theorem implies that VFX should always be zero even in the presence of intense diabatic heating and friction.

A crucial step in the impermeability theorem presented in HM90 is to argue that the movement of ∂V_θ is given by

$$\boldsymbol{U} = \boldsymbol{u} - \frac{H\boldsymbol{\zeta} + \boldsymbol{F}\times\nabla\theta}{\rho Q}\ , \qquad (3)$$

which, after some manipulation, can be put in a different form:

$$\boldsymbol{U} = \boldsymbol{u}_{\theta\perp} + \boldsymbol{u}_\parallel - \frac{H}{\rho Q}\boldsymbol{\zeta}_\parallel - \frac{\boldsymbol{F}\times\nabla\theta}{\rho Q}\ , \qquad (4)$$

where

$$\begin{aligned}\boldsymbol{u}_\parallel &= \boldsymbol{u} - \frac{\boldsymbol{u}\cdot\nabla\theta}{|\nabla\theta|^2}\nabla\theta, \\ \boldsymbol{\zeta}_\parallel &= \boldsymbol{\zeta} - \frac{\boldsymbol{\zeta}\cdot\nabla\theta}{|\nabla\theta|^2}\nabla\theta\ , \quad \text{and} \\ \boldsymbol{u}_{\theta\perp} &= -\frac{\partial\theta/\partial t}{|\nabla\theta|^2}\nabla\theta\ .\end{aligned} \qquad (5)$$

Here, $\perp$ and $\|$ refer to the normal and tangential components with respect to the isentropic surface (see HM90), and the normal component is of interest to us. If Eq. (3) is held, then VFX is obviously nullified as expected. So, the PVS fluxes through the lateral boundaries are the only contributors to the rate of change of PVS within the control volume. For a layer bounded by two closed isentropic surfaces, the lateral fluxes disappear and the total PVS is thus conserved. However, there are two problems with the PVS theorem which may render its application ambiguous: (a) the movement of isentropic surfaces in the presence of general diabatic heating and (b) the development of possible singularity in static stability on the surfaces. They are discussed in the following sections separately.

3. Convectively generated singularity

Let us examine Eq. (3) for the movement of isentropic surfaces and ask what will happen if $Q = 0$ at some points on the surfaces. Needless to say, Eq. (3) will break down at these points, and the isentropic surface across these points will have infinitive velocity! (A similar scenario will appear in the well-mixed boundary layer where intense vertical turbulent mixing occurs, and therefore the isentropic surfaces have no well-defined normal vectors.) So what does this infinity mean? Specifically for our discussion, we consider a layer bounded by two closed isentropic surfaces θ_U and θ_L, and focus on the movement of, say, the lower isentropic surface θ_L on which Q vanishes at some points. In this case, VFX at the θ_L surface is given by

$$\mathrm{VFX_L} = -\int_{\partial\theta_\mathrm{L}} [(\boldsymbol{u}-\boldsymbol{U})\rho Q - H\boldsymbol{\zeta} - \boldsymbol{F}\times\nabla\theta]\cdot\boldsymbol{n}dS\ . \qquad (6)$$

The infinite movement of the θ_L surface due to $Q = 0$ implies that the contributions from the boundary movement, i.e., the term $(\boldsymbol{u}-\boldsymbol{U})\rho Q$ on the rhs of Eq. (6), will vanish at the points where $Q = 0$, so the movement of the θ_L surface at those points is not defined. To see further how $\mathrm{VFX_L}$ looks at these points, let us assume that Q vanishes over a domain Ω on an isentropic surface θ_L that is defined as $\Omega = \{q \in \theta_\mathrm{L}\colon Q(q) = 0\}$. Because of the validity of Eq. (3) $\forall q \notin \Omega$, $\mathrm{VFX_L}$ can be rewritten as

$$\mathrm{VFX_L} = \int_\Omega (H\boldsymbol{\zeta})\cdot\boldsymbol{n}dS\ , \qquad (7)$$

where we have used $Q(q) = 0 \forall q \in \Omega$ and Eq. (3) for $\forall q \in \theta_\mathrm{L}/\Omega$, and the frictional term is orthogonal to the θ_L surface. If PVS cannot penetrate isentropic surfaces as stated in HM90, $\mathrm{VFX_L}$ has to equal to zero on Ω. Apparently, if diabatic heating is absent (i.e.,

H=0), $\mathrm{VFX_L}$ in Eq. (7) [and Eq. (6)] become nullified, thus confirming the impermeability of the PVS across isentropic surfaces. If Q is different from zero at any points, the movement of isentropic surfaces can be defined by Eq. (4), thus nullifying the rhs of Eq. (6) as well.

However, $\mathrm{VFX_L}$ does not necessarily vanish in the presence of arbitrary diabatic heating, given only $Q = 0$ on Ω. Consider, for example, a flat isentropic surface such that $\boldsymbol{n} = (0, 0, 1)$. In this case, the only possibility for $\mathrm{VFX_L}$ to vanish on Ω in the presence of diabatic heating is to require $\zeta_z(q) = 0 \forall q \in \Omega$. Clearly, this requirement is too strong. Indeed, from

$$Q = \frac{1}{\rho}\zeta_z\frac{\partial\theta}{\partial z} = 0,$$

we can have either $\zeta_z = 0$ or $\partial\theta/\partial z = 0$ on Ω. Then $\zeta_z(q) = 0 \forall q \in \Omega$ is just one of many possible ways, and $\mathrm{VFX_L}$ is therefore not necessarily nullified on Ω.

To illustrate the above points, Fig. 1 shows vertical cross sections of PV and θ from a cloud-resolving simulation of Tropical Storm Eugene (2005) using the Weather Research and Forecast (WRF) model with the finest grid size of 1.33 km (see Kieu and Zhang, 2008). One can see from Fig. 1a nearly dry adiabatic lapse rates of the 355 K surface that are collocated with the places where PV is annihilated as a consequence of intense latent heating and vertical motion. Apparently, any location with $\partial\theta/\partial z = 0$ indicates that PVS is penetrating the isentropic surface effectively. As the 355 K surface evolves with time, one can see from Fig. 1b an even more troublesome picture with the formation of 355 K bubbles around the surface associated with deep pulses of moist convection. Even though the emergence of these bubbles occurs within a very short transient period (in both numerical simulations and reality), this indicates that there is nothing, in principle, to prevent the collapse of normal vectors along isentropic surfaces in the presence of diabatic heating.

In the real atmosphere, the neutrality condition $\partial\theta/\partial z$= 0, which may result from adiabatic cooling above diabatic heating associated with organized mesoscale convective systems, is fairly common and has been captured by the conventional upper-air network. One example was given by Fritsch and Maddox (1981) who showed the near neutrality condition over a mesoscale high pressure area between 300 hPa and 150 hPa (see their Figs. 23 and 24). In this case, deep convection produced a layer of neutrality in the lower stratosphere. Similarly, an observational study by Johnson et al. (1990) captured significant descending motion at the top of stratiform clouds in a 2–3-km layer near the tropopause as a result of strong radia-

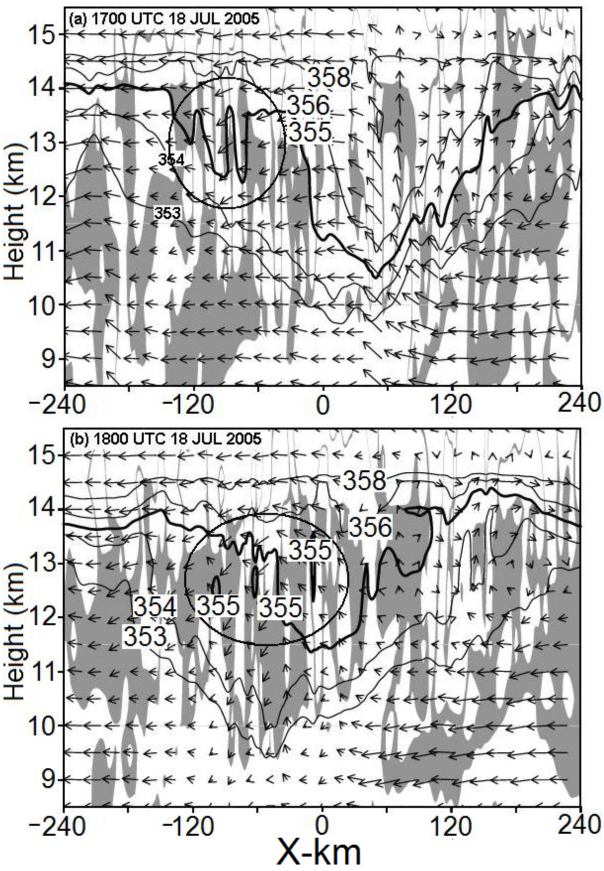

Fig. 1. West-east vertical cross sections of potential temperature (contoured), and PV (shaded within the range of −0.5 PVU< PV <0.5 PVU, 1 PVU=10^{-6} K $\mathrm{m^2}$ $\mathrm{kg^{-1}}$ $\mathrm{s^{-1}}$) at two consecutive snapshots from the 38–39 h cloud-resolving simulations of Tropical Storm Eugene (2005) with the finest grid size of 1.33 km. Superimposed is the in-plane flow vectors. The circle in panel (a) denotes the location at which both Q=0 and $\partial\theta/\partial z$=0 would develop at the later time as shown in panel (b).

tive cooling, estimated at a rate of 0.5°C $\mathrm{h^{-1}}$. [Using a one-dimensional coupled cloud-radiation model, Chen and Cotton (1987) found that the long-wave radiative cooling could be as large as 8°C $\mathrm{h^{-1}}$–10°C $\mathrm{h^{-1}}$ in a layer of ∼100 m near the top of the marine stratocumulus clouds.] Clearly, this descending motion resulted from the local destabilization of the atmosphere by radiative cooling, which would likely lead to the vertical exchange of PVS across isentropic surfaces.

While the breaks in isentropic surfaces caused by the diabatic heating (or cooling) will be quickly smoothed out by 3D flows to stabilize the atmosphere, these two examples show that such breaks are unavoidable in the presence of diabatic heating (or cooling). In HM90 the theorem was carefully limited to a stable atmosphere to ensure that $\partial\theta/\partial z$ >0, but such a constraint may be too strong such that it would eliminate some convective motion in the atmosphere. As discussed above, the presence of a general heating source could lead to a situation in which $\partial\theta/\partial z = 0$, thus violating the *priori* assumption of stably stratified atmosphere in the theorem. We should mention that, while the singularity associated with $\partial\theta/\partial z = 0$ is examined for idealized flat isentropic surfaces with $\boldsymbol{n}$= (0, 0, 1), its implications can be extended also to the PVS theorem defined in the isentropic or pressure coordinates in which only the vertical component of PV is considered. Apparently, at any point where the condition $\partial\theta/\partial z$= 0 is met, PVS will be allowed to penetrate isentropic surfaces.

The neutrality discussed here is just one simple example of a general issue with the PVS theorem, which relies critically on the concept of isentropic surfaces. In the presence of an arbitrary heating profile or within a well-mixed layer, normal vectors $\boldsymbol{n}$ may not be well defined at some points on an isentropic surface, even if this surface is perfectly smooth at the initial time. Thus, the concept of isentropic surfaces becomes ambiguous, and it has to be replaced by the concept of an isentropic layer. Apparently, there are now an infinite number of isentropic surfaces with the same θ within the isentropic layer, and the surface integrals in Eq. (6) will lose all of their physical meaning. Such a peculiar property of isentropic surfaces is rooted in the fact that the potential temperature, or more precisely temperature, is not a purely geometrical quantity that can be defined strictly as "a surface" in a coordinate system. Because temperature is a statistical measure of the mean kinetic energy of molecules at an equilibrium state, it is hard to perceive the concept of "isentropic surface" in the geometrical sense. It follows that PVS is not exactly a kinematic object as discussed in HM90 and Vallis (2005), and the PVS theorem should be understood in a statistical sense rather than being considered a geometrical theorem. Because of this, we conclude that the impermeability of isentropic surfaces by PVS is, at most, valid in the absence of diabatic heating.

4. Ill-defined movement of isentropic surfaces

While the above-mentioned singularity is sufficient to invalidate the PVS theorem in the presence of intense diabatic heating, the extent to which Eq. (3) or (4) can be used to estimate the movement of isentropic surfaces is worth examining. Clearly, Eq. (3) states that the movement of an isentropic surface is only determined by the data on that surface initially. As discussed in section 3, however, the movement of the isentropic surface depends on how the diabatic heating is distributed not only on the surface but also on all other neighboring surfaces. Therefore, the hypothetical velocity $\boldsymbol{U}$ of an isentropic surface should only make sense between two different instants of time rather than instantaneously. But even in this case, there are many ways to define $\boldsymbol{U}$. An example is given in Fig. 2a, which shows that $\boldsymbol{U}$ can take any direction between two isentropic surfaces. The only deterministic component is the orthogonal component that maps the points on the initial surface to the moving surface at a later time. Of course, $\boldsymbol{U}$, given by Eq. (3), is just one of the many possibilities, and it is only determined uniquely from the requirement that VFX has to vanish. In the worst-case scenario, where the isentropic surface at the later time is distorted strongly due to diabatic heating, the movement of the isentropic surface at these points cannot even be defined (Fig. 2b). While the latter indefinite ways of the movement of isentropic surfaces are rare, it could indeed occur sometimes, as shown in the preceding section. In this section, let us assume that no such distortion occurs during the movement of isentropic surfaces, and then examine to what extent Eq. (3) can be used to describe the movement of the isentropic surface.

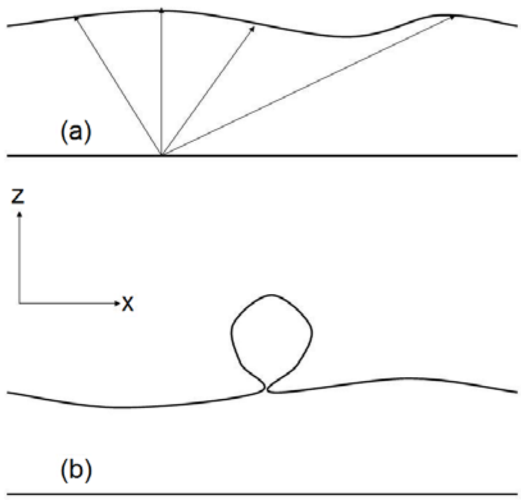

Fig. 2. Schematics of a moving isentropic surface for (a) a regular mapping; and (b) an ill-defined mapping due to the presence of diabatic heating at a later time. Arrows in (a) indicate possible directions of the fictitious velocity of the isentropic surface.

To this end, consider an idealized stable atmosphere with flat isentropic surfaces as shown in Fig. 3, and assume that both the heating function $H(z)$ and vertical motion $w(z)$ do not vary with time. Let us focus on the vertical movement of one specific isentropic surface (say θ=300 K) in this idealized atmosphere, which, according to Eq. (4), is given simply by

$$\boldsymbol{U}(z_{\rm i}) \equiv \begin{pmatrix} 0 \\ 0 \\ W(z_{\rm i}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ w - H/\Gamma \end{pmatrix} , \qquad (8)$$

where $\Gamma(z,t) \equiv \partial\theta(z,t)/\partial z$ is the static stability, $z_{\rm i}$ is the initial position of the 300-K surface, and $W(z_{\rm i})$ is the vertical motion of the isentropic surface [which differs from the vertical motion $w(z_{\rm i})$ of the parcels on the surface at $z = z_{\rm i}$ due to the existence of the diabatic heating source $H(z_{\rm i})$]. Note that in deriving Eq. (8), the local change of θ in the numerator of $\boldsymbol{u}_{\theta\perp}$ has been replaced by the flow field w corrected by the contribution from diabatic heating. In addition, because of diabatic heating, Γ will be a function of both z and t. Given the vertical motion of the 300-K surface in Eq. (8), the "predicted" displacement up to the second order after δt is

$$\delta z_{\rm HM} = W\delta t \approx \left[w(z_{\rm i}) - \frac{H(z_{\rm i})}{\Gamma(z_{\rm i})}\right]\delta t + \frac{H}{\Gamma^2}\frac{d\Gamma}{dt}\delta t^2 + O(\delta t^3) , \qquad (9)$$

where the temporal changes of Γ are taken into account.

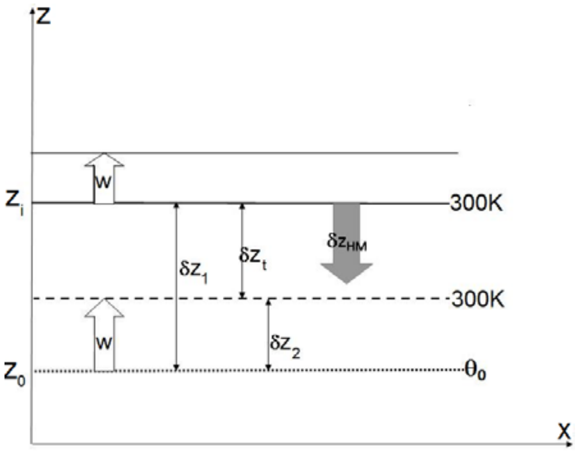

Fig. 3. Vertical movement of the θ=300-K surface after a time interval δt in the presence of diabatic heating $H(z)$. Thick and thin solid lines denote the initial and final locations of the 300-K surface, respectively, without diabatic heating. Dotted lines denotes a θ-surface at t=0 that will develop into a new θ=300 K-surface (shown by thick dashed lines) due to diabatic heating. The hollow arrows denote the flow field w, and the shaded arrow denotes the displacement of the 300 K-surface as predicted by Eq. (4). See the text for various vertical increments in δz.

We next calculate the true distance $\delta z_{\rm t}$ between the initial and final positions of the 300-K surface after the given heating occurs for a time interval δt can be calculated and then compared to the distance $\delta z_{\rm HM}$. To calculate $\delta z_{\rm t}$, we note first that the points on the final 300-K surface (dashed lines) have little to do with those on the initial 300-K surface (i.e., the thick solid line in Fig. 3), but they must stem from a different isentropic surface (e.g., θ_0) at $t = 0$ (dotted lines), depending on the distribution of $\theta(z)$ and $H(z)$. This isentropic surface can be determined from the thermodynamic equation as follows:

$$\delta\theta = 300 - \theta_0 = H(z_{\rm i} - \frac{\delta z_1}{2})\delta t , \qquad (10)$$

where $z_{\rm i}$ is the initial location of the 300-K surface at t= 0 and δz_1 is the distance between 300-K and θ_0 surfaces at t=0 (see Fig. 3). Note that the effects of vertical advection are included in this system moving with the isentropic surface. Given the location of θ_0, the distance δz_1 between the θ_0-surface and 300-K surface at t=0 can be estimated as follows:

$$\delta\theta = \frac{\partial\theta}{\partial z}\delta z_1 = \Gamma(z_{\rm i} - \frac{\delta z_1}{2})\delta z_1 , \qquad (11)$$

where Γ is evaluated at $z = z_{\rm i} - \delta z_1/2$. The distance δz_2 that the θ_0-surface has traveled as a result of diabatic heating during the interval δt is given by

$$\delta z_2 = w(z_{\rm i} - \frac{\delta z_1}{2})\delta t , \qquad (12)$$

where $w(z_0)$ is the ascending motion of the θ_0 surface at $t = 0$. Expanding $w(z_0)$ and $\Gamma(z_0 + \delta z_1/2)$ in the vicinity of $z = z_{\rm i}$, and using Eqs. (10) and (11) to eliminate δz_1, the actual displacement of the 300-K surface $\delta z_{\rm t}$ is now easily computed as $\delta z_{\rm t} = \delta z_2 - \delta z_1$:

$$\delta z_{\rm t} = \delta z_2 - \delta z_1 \approx \left[w(z_{\rm i}) - \frac{H(z_{\rm i})}{\Gamma}\right]\delta t + \left[\frac{dw}{dz} + \frac{H'}{\Gamma} - \frac{H\Gamma'}{\Gamma^2}\right]\frac{H}{2\Gamma}\Big|_{z_{\rm i}}\delta t^2 + O(\delta t^3) \quad , \qquad (13)$$

where the prime denotes a derivative with respect to z. Apparently, $\delta z_{\rm t}$ is the same as $\delta z_{\rm HM}$ for our idealized situation only up to the order $O(\delta t)$. At higher orders, some discrepancies will arise[a].

This second-order difference may seem to imply that the difference is small and can be neglected, or it may be tempting to take $\delta t \to 0$ such that $\boldsymbol{U}$ (or W) can be determined rigorously. Note, however, that this difference reveals conceptually an important eccentricity in defining the movement of isentropic surfaces given by Eq. (4). First, in essence, Eq. (12)

[a]We thank Dr. John Nielsen-Gammon of Texas A&M University for pointing out that a higher-order precision in the calculation of δz_1 can eliminate the first-order discrepancies in Eq. (13). However, the potential singularity due to the vanishing of Γ still exists in Eq. (13), which prohibits an arbitrary Taylor expansion.

indicates that the vertical movement of any isentropic surface in the presence of diabatic heating will depend not only on the information on this surface but also on the other nearby isentropic surfaces, no matter how small δt is. Second, the singularity discussed in the preceding section shows that there are points where $\boldsymbol{U}$ cannot be defined. So, the vertical displacement $\delta z_{\rm t}$ may not possess local behaviors that are smooth enough to allow for taking the limit of $\delta t \rightarrow 0$. Thus, the velocity $\boldsymbol{U}$ given by Eq. (4) [or W in Eq. (8)] does not determine the correct displacement of the isentropic surface, and VFX can take any value. The key point to remember here is that this hypothetical velocity $\boldsymbol{U}$ (or W) is not related to any real motion, but it is defined for a moving isentropic surface only at two different instants of time rather than instantaneously.

Notably, the penetration of PVS across isentropic surfaces does not contradict result presented in HM90 that the volume integration of PVS vanishes for a closed volume bounded by two isentropic surfaces around the globe. The nullity of such a volume integration just implies that the PVS fluxes at the lower isentropic surface are proportional to that at the upper surface (minus the ratio of potential temperatures at the two surfaces), or the PVS fluxes at different locations on a surface are canceled out. For a finite volume, the PVS fluxes are allowed to penetrate across any isentropic surface in the presence of arbitrary diabatic heating.

5. Concluding remarks

In this paper, the PVS impermeability theorem of HM90 was evaluated by examining the two crucial issues involved in developing the theorem: the movement of isentropic surfaces and the singularity of static stability in the presence of diabatic heating. This was done by considering the volume integration of PVS for a finite volume bounded by two isentropic surfaces with diabatic heating. Results show that the movement of isentropic surfaces theorem presented in HM90 is potentially ill-defined in the presence of general diabatic heating.

We emphasized that the movement of an isentropic surface depends not only on the information on this surface but also on the nearby isentropic surfaces. Breaks of isentropic surfaces associated with convectively or radiatively generated singularities in static stability tend to prohibit the well-defined movement of the isentropic surfaces. In particular, we have shown through a cloud-resolving simulation that the presence of a diabatic heating source could eventually violate the assumption of "stably stratified atmosphere" and could invalidate the impermeability theorem. Therefore, we conclude that the impermeability theorem of PVS presented in HM90 can only be held for the stably stratified atmosphere in the absence of diabatic heating, and that PVS can penetrate across any isentropic surface when diabatic (latent or radiative) heating is included. A similar conclusion may be applied to the well-mixed planetary boundary layer.

Finally, it should be pointed out that these conclusions are more suitable for local applications. This implies that the impermeability principle of PVS may be still applicable in the presence of intense diabatic heating when a deep layer, i.e., a layer that is much deeper than the depth of its generated neutral layer, is considered. For example, in the case of Fig. 1b, the impermeability principle may be valid in the isentropic depth between $\theta = 358$ and 353 K surfaces with a positive bulk lapse rate of the potential temperature.

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REFERENCES

Chen, C., and W. R. Cotton, 1987: The physics of the marine stratocumulus-capped mixed layer. *J. Atmos. Sci.*, **44**, 2951–2977.

Danielsen, E. F., 1990: In defense of Ertel's potential vorticity and its general applicability as a meteorological tracer. *J. Atmos. Sci.*, **47**, 2013–2020.

Davis, C. A., and K. Emanuel, 1991: Potential vorticity diagnostics of cyclogenesis. *Mon. Wea. Rev.,* **119**, 424–439.

Egger, J., 2008: Piecewise potential vorticity inversion: Elementary tests. *J. Atmos. Sci.,* **65**, 2015–2024.

Fritsch, J. M., and R. A. Maddox, 1981: Convectively driven mesoscale weather systems aloft. Part I: Observations. *J. Appl. Meteor.*, **20**, 9–19.

Haynes, P., and M. McIntyre, 1987: On the evolution of vorticity and potential vorticity in the presence of piabatic heating and frictional or other forces. *J. Atmos. Sci.*, **44**, 828–841.

Haynes, P., and M. McIntyre, 1990: On the conservation and impermeability theorems for potential vorticity. *J. Atmos. Sci.*, **47**, 2021–2031.

Hoskins, B. J., M. E. McIntyre, and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Meteor. Soc.,* **111**, 877–946.

Huo, Z.-H., D.-L. Zhang, and J. Gyakum, 1999: The interaction of potential vorticity anomalies in extratropical cyclogenesis. Part I: Static piecewise inversion. *Mon. Wea. Rev,* **127**, 2546–2561.

Johnson, R. H., W. A. Gallus, M. D. Vescio, 1990: Near-tropopause vertical motion within the trailing stratiform region of a midlatitude squall line. *J. Atmos.*

Sci., **47**, 2200–2210.

Kieu, C. Q., and D.-L. Zhang, 2008: Genesis of Tropical Storm Eugene (2005) from merging vortices associated with ITCZ breakdowns. Part I: Observational and modeling analyses. *J. Atmos. Sci.*, **65**, 3419–3439.

Kieu, C. Q., and D.-L. Zhang, 2010: A piecewise potential vorticity inversion algorithm and its application to hurricane inner-core anomalies. *J. Atmos. Sci.*, **67**, 2616–2631.

Vallis, G. K., 2005. *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-scale Circulation*, Cambridge University Press, 745pp.

Zhang, D.-L., W. Cheng, and J. Gyakum 2002: The impact of various potential vorticity anomalies on multiple frontal cyclogenesis events. *Quart. J. Royal Meteor. Soc.*, **128**, 1847–1878.

Zhang, D.-L., and C. Q. Kieu, 2006: Potential vorticity diagnosis of a simulated hurricane. Part II: Quasi-balanced contributions to forced secondary circulations. *J. Atmos. Sci.*, **63**, 2898–2914.